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## Polytropic Stellar Models:

For stars in hydrostatic equilibrium the pressure gradient follows (we assume spherical symmetry):

$$\frac{dP}{dr} = -\frac{d\Phi}{dr}\rho$$

Here $\Phi$ is the gravitational potential and obeys Poisson's equation (recall that we use Newtontian gravity):

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\rho$$

In general, the mass density is a function of pressure $P$ and temperature $T$: $\rho = \rho(P,T)$. If $\rho$ does not depend on $T$, then the two equations in above can be solved to find $\rho$ (hence $P$) as a function of $r$, provided that $\rho = \rho(P)$ is known.

Here we consider cases where there is a simple relation between $\rho$ and $P$ of the form:

$$P = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}}$$

This is called "Polytropic relation". Here $K$, $\gamma$, $n$ are the polytropic constant, polytropic exponent, and polytropic index respectively.

In some cases such a relation can be derived from first principles and $K, \gamma$ (equivalently $K, n$) can be explicitly calculated. For example, for a degenerate electron gas one finds $P = K\rho^{\frac{5}{3}}$ (non-relativistic case) and $P = K\rho^{\frac{4}{3}}$ (relativistic case), and $K$ can be calculated in each case.

In some cases, one can use a polytropic relation phenomenologically by choosing $K, n$ to fit the relation between $\rho$ and $P$.

This will greatly simplify the equation for hydrostatic equilibrium and Poisson's equation, which allows us to find solutions for the star.

Let us consider two examples of the latter:

(1) Consider a completely convective star. To a very good approximation we have:

$$\nabla \equiv \frac{d\ln T}{d\ln P} = \nabla_{ad}$$

Assuming a monatomic gas, we find $\nabla_{ad} = \frac{\gamma_{ad}}{\gamma_{ad}-1} = \frac{2}{5}$. This results in $T \propto P^{\frac{5}{2}}$. For an ideal gas we have $P \propto \rho T$ from the equation of state. We therefore find:

$$P \propto \rho T \propto \rho P^{\frac{5}{2}} \Rightarrow \rho \propto P^{\frac{5}{3}} \Rightarrow \boxed{P = K\rho^{\frac{5}{3}}}$$

This is a polytropic relation with $\gamma = \frac{5}{3}$ ($n = \frac{3}{2}$). It is important to note that the polytropic constant cannot be calculated from first principles. It is a free fitting parameter that should be fixed by other considerations (as we will see shortly), and it varies from one star to another.

(2) Next consider Sun, which has a radiative core. The total

Pressure is given by:

$$P = \underbrace{\frac{1}{3} a T^4}_{\text{radiation}} + \underbrace{\frac{\rho}{\nu m_u} k_B T}_{\text{matter}}$$ ($\nu$: mean molecular weight, $m_u$: atomic mass unit)

If we assume $\frac{P_{rad}}{P} = 1-\beta$, with $\beta$ being a constant), we can write $P$ as follows:

$$P = \left(\frac{3 k_B^4}{a \nu^4 m_u^4}\right)^{1/3} \left(\frac{1-\beta}{\beta}\right)^{1/3} \rho^{4/3} \Rightarrow \boxed{P = K \rho^{4/3}}$$

This is a polytropic relation with $\gamma = \frac{4}{3}$ $(n=3)$. Again note that $K$ cannot be calculated from first principles. (As a side note, the above polytropic relation is just Eddington's standard model. He found that equations for stellar evolution can be solved very simply by making the assumption that $\frac{\kappa L_r}{M_r}$ = constant. This is equivalent to $\beta$ being constant.)

## Lane-Emden Equation:

Assuming a polytropic relation $P = K\rho^{1+\frac{1}{n}}$, we find from the equation of hydrostatic equilibrium:

$$\frac{d\Phi}{dr} = -\gamma K \rho^{\gamma-2} \frac{d\rho}{dr} \Rightarrow \rho = \left(\frac{-\Phi}{(n+1)K}\right)^n \quad (\Phi = 0 \text{ at the surface})$$

Replacing this expression in Poisson's equation, and after making the following change of variables:

$$z = Ar \;,\quad A^2 = \frac{4\pi G}{(n+1)^n K^n}(-\Phi_c)^{n-1} = \frac{4\pi G}{(n+1)K}\rho_c^{\frac{n-1}{n}} \;,\quad w = \frac{\Phi}{\Phi_c} = \left(\frac{\rho}{\rho_c}\right)^{\frac{1}{n}}$$

($\rho_c, \Phi_c$ being the mass density and gravitational potential at the center $z = r = 0$ respectively), we find the following equation:

$$\boxed{\frac{1}{z^2}\frac{d}{dz}\left(z^2 \frac{dw}{dz}\right) + w^n = 0}$$

This is the famous Lane-Emden equation. We are interested is solutions that are finite at the center.

In order for $\frac{d^2w}{dz^2}$ not to blow up at $z=0$, we then have to require $\frac{dw}{dz}=0$ at $z=0$. For finite solutions, we also have $w(0)=1$.

The solutions to the Lane-Emden equation, with boundary conditions $w(0)=1$ and $w'(0)=0$, are known for $n=0,1,5$ cases. The equation must be solved numerically in the other cases.

For the star to be finite, we need $w(z_n)=0$ at finite $z_n$. This happens to be the case for $n<5$. For $n\geq 5$ the radius of the star will be infinite.

Application to Stars:

Lets construct polytropic models with an index $n<5$ for stars. One has to distinguish between two cases: $K$ as a free parameter, and $K$ being fixed.

(a) K a free parameter: Using the Lane-Emden equation, we have:

$$M_r = \int_0^r 4\pi r^2 \rho \, dr = 4\pi \rho_c \int_0^r \omega^n r^2 \, dr = 4\pi \rho_c \left(\frac{r}{z}\right)^3 \int_0^z \omega^n z^2 dz =$$

$$4\pi \rho_c r^3 \left(-\frac{1}{z}\frac{d\omega}{dz}\right)$$

This implies that:

$$\boxed{M = 4\pi \rho_c R^3 \left(-\frac{1}{z}\frac{d\omega}{dz}\right)_{z=z_n}} \quad *$$

The average density of the star $\bar{\rho}$ obeys the following equation:

$$\frac{\bar{\rho}}{\rho_c} = \left(\frac{-3}{z}\frac{d\omega}{dz}\right)_{z=z_n}$$

If mass M and radius R of the star are specified, we can easily find $\bar{\rho}$. The polytropic index n is chosen appropriately (on physical grounds). For a given index $n<5$, we can find $z_n$ and $\omega'(z_n)$. Hence, $\rho_c$ will also be known.

The polytropic constant K can now be fixed by noticing that $A = \frac{z_n}{R}$. Having found $z_n$ (with R specified), we can find A. Recall that:

$$A = \frac{4\pi G}{(n+1)K} \rho_c^{\frac{n-1}{n}}$$

The only unknow parameter in this expression is K, which can now be fixed.

Once K is known, we can find $P_{(r)}$ as a function of r. Therefore, the whole mechanical structure of the star is now determined. One can also find $T_{(r)}$ by using the equation of state for an ideal gas.

As an example, lets construct a polytropic model for Sun. As mentioned before, $n=3$ is a justified polytropic index in this case. For $n=3$, it turns out that:

$$\boxed{z_3 = 6.897 \,, \quad \frac{\bar{\rho}}{\rho_c} = 54.18}$$

For Sun $M = 1.989 \times 10^{33}$ g and $R = 6.96 \times 10^{10}$ cm. This results in an average density $\bar{\rho} = 1.41$ g $cm^{-3}$, and hence $\rho_c = 76.39$ g $cm^{-3}$.

We also find:

$$A = \frac{z_3}{R} = 9.91 \times 10^{-11}$$

This will fix the polytropic constant:

$$\boxed{K = 3.85 \times 10^{14}}$$

We can now find the central pressure $P_c$:

$$\boxed{P_c = 1.24 \times 10^{17} \text{ dyn cm}^{-2}}$$

Using the ideal gas equation $P = \frac{\rho}{\nu m_u} k_B T$, we can also find $T_c$. Assuming a chemical composition with $X \simeq 0.7$ and $Y \simeq 0.3$, we have $\nu \simeq 0.62$, which results in:

$$\boxed{T_c = 1.2 \times 10^{7} \text{ K}}$$

The values obtained for $P_c$, $T_c$ from a polytropic estimate are considerably closer to what is found from a proper numerical solution of the full set of stellar equations than our crude estimate at the beginning of this course.

(b) K a fixed parameter. The situation will be different if K is a fixed parameter that can be calculated from first principles. As an example, consider a degenerate electron gas in the non-relativistic regime. In this case one finds a polytropic relation with $n = \frac{3}{2}$:

$$P = K \rho^{\frac{5}{3}} \quad ; \quad K = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m_e} \frac{1}{(\nu_e m_U)^{\frac{5}{3}}}$$

Here $\nu_e m_U$ is the average particle masses per (free) electron.

For a given chemical composition $\nu_e$ (hence K) is determined.

For a polytropic model with index $n$ and fixed constant K the central density $\rho_c$ and A are related through:

$$A^2 = \frac{4\pi G}{(n+1) K} \rho_c^{\frac{n-1}{n}}$$

Recall that $R = \frac{z_n}{A}$. Since the value of $z_n$ can be found (numerically), we then have:

$$R \sim \rho_c^{\frac{1-n}{2n}}$$

On the other hand:

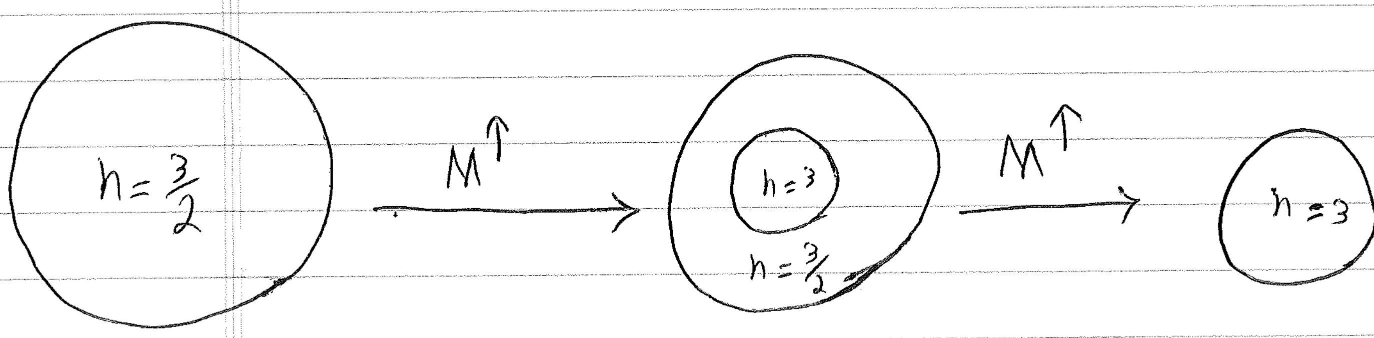
$$M = 4\pi \rho_c R^3 \left(-\frac{1}{z}\frac{dw}{dz}\right)_{z=z_n} \Rightarrow M = \underbrace{4\pi \left(-\frac{w'}{z}\right)_{z=z_n} z_n^3 \left(\frac{n+1}{4\pi G}\right)^{3/2} K^{3/2}}_{\text{Const.}} \rho_c^{\frac{3-n}{2n}}$$

Eliminating $\rho_c$ between the expressions for $R$, $M$ we arrive at:

$$\boxed{R \sim M^{\frac{1-n}{3-n}}}$$

Therefore, for a chosen polytropic index $n$ and fixed polytropic constant $K$, the radius and mass of the star are related to each other. We have a one-dimensional manifold of models in this case, $M$ or $R$, unlike the case with $K$ being a free prameter where there was a two-dimensional manifold of models ($M$ and $R$).

A surprising aspect of the Polytropic model with $n=\frac{3}{2}$ and fixed K is that the radius R decreases with increasing the mass M. This signals the break up of the validity of $n=\frac{3}{2}$ as an appropriate polytropic index. Since $\rho_c \propto M^2$ in this case, the central density increases with increasing M. The Fermi energy $E_f \propto \rho^{\frac{2}{3}}$, and hence $E_f$ at the center goes up as M increases. In consequence, there will be a transition from non-relativistic to relativistic regime. The suitable polytropic index for a relativistic degenerate electron gas is $n=3$ ($\gamma=\frac{4}{3}$). The transition happens first at the core where density is higher. A star consisting of degenerate matter can be thought of having a relativistic core surrounded by a non-relativistic envelope. The relativistic core occurs at $\rho_c \gtrsim 10^6\ g\ cm^{-3}$, and it gradually encompasses as $\rho_c$ increases further. Eventually, a polytropic model with $n=3$ describes the whole star. This is how Chandrasekhar constructed his first white dwarf model. Schematically:




## Chandrasekhar's Mass Limit:

As we saw, for a polytropic model with index $n$ and fixed $K$, we have:

$$M = 4\pi \left(-\frac{w'}{z}\right)_{z=z_n} z_n^3 \left(\frac{n+1}{4\pi G}\right)^{3/2} K^{3/2} \rho_c^{\frac{3-n}{2n}}$$

For $n=3$ (i.e. relativistic degenerate electron gas) $M$ becomes a constant:

$$\boxed{M = 4\pi \left(-\frac{w'}{z}\right)_{z_3} z_3^3 \left(\frac{K}{\pi G}\right)^{3/2}}$$

This is called the Chandrasekhar's mass, which after inserting proper numerical values becomes:

$$\boxed{M_{Ch} = \frac{5.836}{\mu_e^2} M_\odot}$$

This limiting mass implies that there will be no polytropic solutions for $M > M_{Ch}$. Equivalently, no stable star consisting of degenerate matter can exist with $M$ beyond $M_{Ch}$.

Stars whose equation of state is dominated by a degenerate electron gas are called white dwarf. They are formed of material where all the Hydrogen is converted to Helium (mainly), Carbon, and Oxygen. In this case $\mu_e = 2$ (two nucleon per electron), which results in:

$$\boxed{M_{Ch} = 1.46\, M_\odot}$$

No white dwarf has been found that exceeds this mass.

## Gravitational and Total Energy for Polytropes:

The gravitational energy, by definition, is:

$$\Omega = -G\int_0^M \frac{M_r}{r}\,dM_r$$

After integration by part it becomes:

$$\Omega = -\frac{1}{2}\frac{GM^2}{R} - \frac{1}{2}\int_0^R \frac{GM_r^2}{r^2}\,dr$$

Using the relation $\frac{d\Phi}{dr} = \frac{GM_r}{r^2}$ and performing another integration by part,

we find:

$$\Omega = -\frac{1}{2}\frac{GM^2}{R} + \frac{1}{2}\int_0^M \Phi\,dM_r \qquad \text{(recall that } \Phi = 0 \text{ at the surface)}$$

For a polytropic model:

$$\Phi = -\frac{K\gamma}{\gamma-1}\rho^{\gamma-1} = -\frac{\gamma}{\gamma-1}\frac{P}{\rho} \Rightarrow \Omega = -\frac{1}{2}\frac{GM^2}{R} - \frac{1}{2}\frac{\gamma}{\gamma-1}\int_0^M \frac{P}{\rho}\,dM_r \quad *$$

We can write the second term on the right-hand side of last equality in a more useful form. We start from:

$$\int_0^M 4\pi r^3 \frac{dP}{dM_r}\,dM_r = \int 4\pi r^3\,dP = \underbrace{\left[4\pi r^3 P\right]_0^M}_{0} - \int_0^M 12\pi r^2 P \frac{dr}{dM_r}\,dM_r$$

From equation of hydrostatic equilibrium, we have $\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4}$. Hence:

$$\int_0^M 4\pi r^3 \frac{dP}{dM_r} dM_r = -\int_0^M \frac{GM_r}{r} dM_r$$

Also:

$$\int_0^M 12\pi r^2 P \frac{dr}{dM_r} dM_r = \int_0^M \frac{3P}{\rho} dM_r$$

We therefore find:

$$\int_0^M \frac{P}{\rho} dM_r = -\frac{1}{3}\int_0^M \frac{GM_r}{r} dM_r = -\frac{1}{3}\Omega \quad **$$

Equations * and ** lead to:

$$\Omega = -\frac{1}{2}\frac{GM^2}{R} + \frac{1}{6}(n+1)\Omega \Rightarrow \boxed{\Omega = \frac{-3}{5-n}\frac{GM^2}{R}}$$

Here $n = \frac{1}{\gamma - 1}$ is the polytropic index. For $n < 5$, such that $R$ is finite, we see that $\Omega < 0$ (as expected).

The total energy, which is the sum of gravitational potential energy and internal energy, can be found by using Virial theorem. It states that (see pag ⑤, lecture 1):

$$\int_0^M \frac{3P}{\rho}\, dM_r + \Omega = 0$$

For an ideal gas,

$P = (\gamma_{ad} - 1)\rho u$ (u: internal energy per unit mass)

Thus:

$$U = \int_0^M u\, dM_r = \frac{-1}{3(\gamma_{ad}-1)}\Omega \Rightarrow \boxed{W = U + \Omega = \frac{3}{5-n}\,\frac{4-3\gamma_{ad}}{3\gamma_{ad}-3}\,\frac{GM^2}{R}}$$

assuming all of the internal energy is kinetic

Note that $W < 0$ for $\gamma_{ad} > \frac{4}{3}$, which we saw before.